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Static perfect fluid cylinders—a new approach to Kramer's equations

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Abstract. Using Kramer's formulation of static cylindrically symmetric perfect fluid solutions, we propose a fairly general integration procedure which yields existing solutions as special cases. Foremost, we reduce the problem of finding the cylindrically symmetric analogue of the Schwarzchild interior solution to that of solving a nonlinear second-order ordinary differential equation for the generating function. Although we cannot solve this equation exactly, we provide an asymptotic analysis of its solution under the assumption that the pressure is very small within the source.

1. Introduction

Vacuum solutions with cylindrical symmetries abound in the literature. But realistic sources producing them are scarce. Over the years, many authors have proposed various formulations of the cylindrically symmetric Einstein field equations that simplify the search of sources. Amongst these formulations, those of Evans [1], Kramer [2] and Philbin [3] are particularly remarkable thanks to their simplicity and implications (for instance, the derivation of physically realistic sources for the Levi-Civita vacuum solution). In Evans' scheme, one must choose an appropriate generating function and then solve a scalar linear second-order ODE. For its part, Philbin reduced the field equation to a scalar linear first-order ODE that can be solved for a convenient generating function. Using a nonlocal change of variables, Kramer mapped the field equations to a pair of coupled first-order nonlinear ODEs which can be uncoupled to yield a single nonlinear second-order ODE [5]. We shall adopt this last approach in this paper.

It should be noted that Haggag and Desokey [4] and Haggag [5] also used Kramer's equations in the derivation of their solutions. They assumed an ansatz for the solutions of Kramer's equations.

Here, we shall provide a broader setting for the solution to Kramer's equations. As a result, existing solutions become particular cases of our solution. Further, we propose a solution to the problem of finding a cylindical source with constant density.

2. Field equations: Kramer's formulation

The static cylindrically symmetric metric may be written in Weyl's form as

$$\mathrm{d}s^2 = \mathrm{e}^{2u}\,\mathrm{d}t^2 - \mathrm{e}^{2k-2u}(\mathrm{d}\rho^2 + \mathrm{d}\xi^2) - w^2\mathrm{e}^{-2u}\,\mathrm{d}\phi^2 \tag{1}$$

where the metric functions depend on ρ only and the axis of symmetry is given by $\rho = 0$. For a perfect fluid, the energy momentum tensor is provided by

$$T^{ab} = (\mu + p)U^a U^b - p g_{ab} \tag{2}$$

where μ and p are, respectively, the fluid density and pressure and U^a is the four-velocity of the matter in a comoving frame. The Einstein field equations for a static cylindrically symmetric perfect fluid distribution is then [6]

$$w'' = 2\kappa_0 w p \mathrm{e}^{2k-2u} \tag{3}$$

$$w'' - 2w'k' + 2wu'^2 = 0 \tag{4}$$

$$k'' + u'^2 = \kappa_0 p \mathrm{e}^{2k-2u} \tag{5}$$

$$u'' + \frac{u'w'}{w} = \frac{1}{2}\kappa_0(\mu + 3p)\mathrm{e}^{2k-2u} \tag{6}$$

where a prime denotes differentiation with respect to ρ and κ_0 is the Einstein constant. The equation $T^a_{b;a} = 0$ implies that

$$\frac{\mathrm{d}p}{\mathrm{d}u} + p + \mu = 0. \tag{7}$$

Note that (6) is a consequence of (3)–(5) and (7). Treating u as the new radial coordinate and making the change [2]

$$y = \frac{\mathrm{d}k}{\mathrm{d}u} \qquad z = \frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}u} \tag{8}$$

the field equations become [2]

$$\dot{y} = (1 - yz)(Fy - 2) \qquad \dot{z} = (1 - yz)(Fz - 2) \tag{9}$$

where the overdot indicates differentiation with respect to u and

$$F = \frac{\mu + 3p}{2p}. \tag{10}$$

Interchanging y and z in (9) leaves the equation unchanged. Hence $y \leftrightarrow z$ is a discrete symmetry of (9). For a given equation of state $\mu = \mu(p)$, the pressure $p = p(u)$ is found by solving (7). From (10) it then follows that $F = F(u)$. Once a solution to (9) is known, equations (8) yield the remaining unknowns of the metric. Indeed if u is taken as the new radial coordinate, the metric reads [2, 5]

$$\mathrm{d}s^2 = \mathrm{e}^{2u}\,\mathrm{d}t^2 - \frac{yz - 1}{\kappa_0 p}\,\mathrm{d}u^2 - \mathrm{e}^{2k-2u}\,\mathrm{d}\xi^2 - w^2\mathrm{e}^{-2u}\,\mathrm{d}\phi^2. \tag{11}$$

The energy condition $\mu \geqslant p$ will be satisfied if $F \geqslant 2$ (use (10)). Further, the elementary flatness at the axis which can be taken to be $u = 0$ modulo a translation, implies $y \to y_0 = \text{const} \neq 0$, $z \to \infty$ and $F \to 2/y_0$ as $u \to 0$ [5].

Note that the metric (11) has an apparent singularity at the axis $u = 0$ which can be romoved by the transformation $u \to r^2$ [5]. If the pressure vanishes for $u = u_b$, we must have $yz \to 1$ as $u \to u_b$ in order to get rid of the singularity at the boundary $u = u_b$ [5].

3. Solutions to Kramer's equations

The system (9) can be uncoupled to give [5]

$$(2 - FY)Y\ddot{Y} = (2 - 3FY)\dot{Y}^2 + [4 - 6FY + (4 + 2F^2 - \dot{F})Y^2 - FY^3]\dot{Y} \tag{12}$$

where $Y \in \{y, z\}$. Now assume that $F = F(Y) \neq 2/Y$ ($F = 2/Y$ leads to Kramer's solutions [2]). Then

$$\dot{F} = F'\dot{Y} \tag{13}$$

where this time the prime stands for differentiation with respect to Y. Using (13) in (12) yields

$$(2 - FY)Y\ddot{Y} = (2 - 3FY - F'Y^2)\dot{Y}^2 + [4 - 6FY + (4 + 2F^2)Y^2 - 2FY^3]\dot{Y}. \tag{14}$$

Note that equation (14) is invariant under the translation of u. Hence the assumption made introduces a symmetry. This symmetry hints to the change of variable $Z = \dot{Y}$. In the new variable, equation (14) reads

$$\frac{\mathrm{d}Z}{\mathrm{d}Y} + \frac{F'Y^2 + 3FY - 2}{Y(FY - 2)}Z = \frac{2FY^3 - (4 + 2F^2)Y^2 + 6FY - 4}{Y(FY - 2)}. \tag{15}$$

Equation (15) is a linear first-order ODE which is easily integrated. The solution to (14) is

$$\int \frac{\mathrm{d}Y}{\mathrm{e}^{I(Y)} \int J(Y)\mathrm{e}^{-I(Y)}\,\mathrm{d}Y + C_1\mathrm{e}^{I(Y)}} = u + C_2 \tag{16}$$

where

$$I(Y) = \int \frac{F'Y^2 + 3FY - 2}{Y(FY - 2)}\,\mathrm{d}Y \tag{17}$$

$$J(Y) = \frac{2FY^3 - (4 + 2F^2)Y^2 + 6FY - 4}{Y(FY - 2)}. \tag{18}$$

To the best of our knowledge, the general solution to Kramer's equations (16) is new and has not appeared in the literature before.

4. Some closed-form solutions

Here, we explore some solutions expressible in terms of elementary functions. We also show that existing solution can be derived from (16) as well.

4.1. Haggag's solutions

Haggag [5] obtained solutions of (9) by making the ansatz

$$z = \frac{\mathrm{d}y^2 + cy - b}{y - a}. \tag{19}$$

It is then straightforward to see that his solutions correspond to the choice

$$Y = y \qquad F = 2\frac{(1 - d)y^2 + 2a(d - 1)y + a^2 + ac - b}{(ad + c)y^2 - 2by + ab} \tag{20}$$

in our formulation.

4.2. Philbin's solutions

They correspond to the relation [5]

$$18(3 - 2z^2)(y - z)^2 + s(y - z)(225y - 300z - 100yz^2 + 224z^3) - 5s^2(5 + 5yz - 8z^2)^2 = 0 \tag{21}$$

where s is a constant. Equation (21) can be solved explicitly to obtain $y = f(z)$. Indeed (21) is a quadratic equation for y. Then equation (9) implies that $F = F(z)$. Whence Philbin's solutions are also derivable from (16).

4.3. New solutions

To obtain solutions in terms of elementary functions, F must be chosen so that it is possible to evaluate the integrals appearing in (16).

If we assume that $I(Y) = 0$ in (16) then we easily find that the F that realizes this is

$$F = \frac{1}{Y} + \frac{K}{Y^3} \tag{22}$$

where K is an arbitrary constant. We assume that $K \neq 0$ since $K = 0$ leads to Haggag's solution. If we further set $Y = y$, (16) becomes

$$\int \frac{y^2}{y^4 + C_1 y^2 + K}\,\mathrm{d}y = u + C_2. \tag{23}$$

From (9), we infer that

$$z = \frac{1}{y} + \frac{y^4 + C_1 y + K}{y(y^2 - K)}. \tag{24}$$

From the expression for z, we see that K must be positive to allow the existence of an axis (cf section 2). Then the axis is given by $y = \pm\sqrt{K}$ provided $K \pm C_1 K^{-1/2} + 1 \neq 0$. Indeed we must have $\lim_{y\to\pm\sqrt{K}} z = \infty$. We choose $y = \sqrt{K}$ as the axis. The other choice can be treated likewise.

Using (10), we find that

$$\mu = \frac{-3y^3 + 2y^2 + 2K}{y^3}\,p. \tag{25}$$

By substituting (25) into (7), we get

$$p = p_0 \exp\left[2\int \frac{y^3 - y^2 - K}{y(y^4 + C_1 y^2 + K)}\,\mathrm{d}y\right]. \tag{26}$$

For convenience, it is preferable to rewrite (11) as

$$\mathrm{d}s^2 = \mathrm{e}^{2u(y)}\,\mathrm{d}t^2 - \frac{yz - 1}{\kappa_0 p}\frac{\mathrm{d}y^2}{\dot{y}^2} - \mathrm{e}^{2k(y)-2u(y)}\,\mathrm{d}\xi^2 - w^2(y)\mathrm{e}^{-2u(y)}\,\mathrm{d}\phi^2$$

namely

$$\mathrm{d}s^2 = \mathrm{e}^{2u(y)}\,\mathrm{d}t^2 - \frac{y^4}{\kappa_0 p(y^2 - K)(y^4 + C_1 y^2 + K)}\,\mathrm{d}y^2 - \mathrm{e}^{2k(y)-2u(y)}\,\mathrm{d}\xi^2 - w^2(y)\mathrm{e}^{-2u(y)}\,\mathrm{d}\phi^2. \tag{27}$$

To complete the description of the solution, we need to evaluate the integral occuring in (23). Let $\Delta = C_1^2 - 4K$. Denote the roots of the quadratic $\lambda^2 + C_1\lambda + K = 0$ by λ_1 and λ_2. The only case allowing the existence of a surface of vanishing pressure is the following:

$\lambda_1 = a^2, \lambda_2 = b^2, a > 0, b > 0$:

$$\left(\frac{y-a}{y+a}\right)^{\frac{a}{2(a^2-b^2)}}\left(\frac{y-b}{y+b}\right)^{\frac{b}{2(b^2-a^2)}} = C_2\mathrm{e}^u \tag{28}$$

where C_2 is an arbitrary constant of integration and

$$\mathrm{e}^k = k_0\left(\frac{y-a}{y+a}\right)^{\frac{a^2}{2(a^2-b^2)}}\left(\frac{y-b}{y+b}\right)^{\frac{b^2}{2(b^2-a^2)}} \tag{29}$$

$$w = w_0\left(\frac{y-a}{y+a}\right)^{\frac{1}{2(a^2-b^2)}}\left(\frac{y-b}{y+b}\right)^{\frac{1}{2(b^2-a^2)}}(y^2 - K)^{-1/2} \tag{30}$$

$$p = p_0 y^{-2}(y-a)^{\frac{a^3-a^2-K}{a^2(a^2-b^2)}}(y+a)^{\frac{a^3+a^2+K}{a^2(b^2-a^2)}}(y-b)^{\frac{b^3-b^2-K}{b^2(b^2-a^2)}}(y+b)^{\frac{b^3+b^2+K}{b^2(a^2-b^2)}} \tag{31}$$

where k_0, w_0 and p_0 are constants.

4.4. Matching to the Levi-Civita metric

Now we consider the problem of matching the solutions obtained in the previous section to the vacuum solution. The exterior Levi-Civita metric may be written as [4]

$$\mathrm{d}s^2 = \mathrm{e}^u\,\mathrm{d}t^2 - \frac{n^2(1-m)^2}{m^2}\mathrm{e}^{2(m-1+1/m)u}\,\mathrm{d}u^2 - \mathrm{e}^{2(1-m)u}\,\mathrm{d}\xi^2 - \mathrm{e}^{2(1/m-1)u}\,\mathrm{d}\phi \tag{32}$$

where $\sigma = m/2$ is the mass per unit length in the Newtonian limit and n is a constant. The constants m and n are related to the internal structure of the source.

A metric is matchable to the exterior vacuum one if there is a surface of vanishing pressure (boundary), the metric coefficient as well as their first derivatives are continuous at the boundary and the quantity $G^i_j f_{,i}$ are continuous at the boundary, where G^i_j is the Einstein tensor and $f = 0$ is the equation of the boundary [7, 8]. Note that the continuity of the radial derivative of the radial component of the metric (i.e. $g_{uu,u}$) is not necessary: it is well known that in the interior and exterior Schwarzschild metrics, the derivative of the radial coefficient is not continuous at the boundary.

Since for our solutions the pressure and the density vanish at the boundary, the junction condition $G^i_j f_{,i} = 0$ at the boundary is automatically satisfied.

For the Levi-Civita metric (32) we have $y = m$. Hence for our solutions, the boundary is necessarily given by $y = y_b = m$. Thus, we choose $a = m$. Hence $b = \sqrt{K}/m$. There is a unique boundary provided $\sqrt{K} < a = m < b = \sqrt{K}/m$. This last condition prevents singularities occurring within the source. The regularity at the axis $y = \sqrt{K}$ leads to

$$C_2 = \left(\frac{\sqrt{K}-m}{\sqrt{K}+m}\right)^{\frac{m}{2(m^2-K^2/m^2)}} \left(\frac{\sqrt{K}-\sqrt{K}/m}{\sqrt{K}+\sqrt{K}/m}\right)^{\frac{\sqrt{K}}{2m(K^2/m^2-m^2)}}. \tag{33}$$

Since the y coordinate is singular beyond the boundary ($y = m$), we cannot express the Levi-Civita metric in the (t, y, ξ, ϕ) coordinates. This prevents us from studying the continuity of the metric coefficients as well as their first-order derivatives at the boundary. Hence we could not relate the coefficients k_0, w_0 and p_0 to m and n. This is one of the drawbacks of Kramer's formulation.

5. Cylinder with a constant energy density

If the energy density of the cylindrical distribution is constant, i.e. $\mu = \mu_0 = \text{const}$, we deduce from (10) that

$$p = \frac{\mu_0}{2F-3}. \tag{34}$$

Substituting (34) into (7) and using (16) with $Y = y$ we obtain

$$(1-F)(2F-3)F'' + \left[4F - 5 - \frac{(2y^3-5y^2+6y)F - y^2 - 4}{y(Fy-2)}\right]F'^2 - \frac{(2F-3)(F-1)(3yF-2)}{y(Fy-2)}F' = 0. \tag{35}$$

For the energy condition to be valid, we require that $F \geqslant 2$. Hence the trivial solutions of (35) $F = 1$ and $\frac{3}{2}$ bear no physical significance. Moreover if $y = y_b$ describes the surface of vanishing pressure, we must have [5]

$$\lim_{y\to y_b} F = \infty.$$

Also, if the axis is $y = y_0$, we must have [5]

$$\lim_{y \to y_0} F = 2/y_0.$$

We could not solve (35) subject to the above constraints. Nevertheless, if we assume that F is very large, that is the pressure is very small within the source, (35) simplifies to

$$F'' - \frac{2}{F}F'^2 + \frac{3}{y}F' = 0. \tag{36}$$

Hence

$$F \sim \frac{4}{y^4 - 4C_1 y + 4C_2} \tag{37}$$

with

$$C_1 = \frac{y_b^4 - y_0^4 + 2y_0}{4(y_b - y_0)} \qquad C_2 = \frac{y_0 y_b^4 - y_b y_0^4 + 2y_0 y_b}{4(y_b - y_0)}.$$

Simple calculations show that

$$p \sim \mu_0 \frac{y^4 - 4C_1 y + 4C_2}{8 - 12C_2 + 12C_1 y - 3y_4} \tag{38}$$

$$\dot{y} \sim \frac{2(C_3 - y^4)}{y^3(y^4 - 4C_1 y + 4C_2)} \tag{39}$$

where C_3 is an arbitrary constant. In the calculations below, we assume that $C_3 > 0$. The case $C_3 < 0$ can be treated similarly. By integrating (39) we find

$$u \sim C_1 y - \frac{y^4}{8} - C_1 C_3^{1/4} \arctan(C_3^{-1/4} y) + \frac{C_1 C_3^{1/4}}{2} \ln(C_3^{1/2} - y^2)$$
$$-\tfrac{1}{8}(4C_2 + C_3)\ln(y^4 - C_3) - C_4 \tag{40}$$

where

$$C_4 = C_1 y_0 - \frac{y_0^4}{8} - C_1 C_3^{1/4} \arctan(C_3^{-1/4} y_0)$$
$$+\frac{C_1 C_3^{1/4}}{2} \ln(C_3^{1/2} - y_0^2) - \frac{1}{8}(4C_2 + C_3)\ln(y_0^4 - C_3)$$

$$2(k - k_0) \sim -(4C_2 + C_3)y + 2C_1 y^2 - \frac{y^5}{5} + \frac{1}{2}C_3^{1/4}(4C_2 + C_3)\arctan(C_3^{-1/4} y)$$
$$+\tfrac{1}{4}(-4C_2 C_3^{1/4} + 4C_1 C_3^{1/2} - C_3^{5/4})\ln(C_3^{1/2} - y^2) - C_1 C_3^{1/2}\ln(y^2 + C_3^{1/2}) \tag{41}$$

where k_0 is an arbitrary constant and

$$\ln\left(\frac{w}{w_0}\right) \sim -\frac{y^2}{3} + \frac{4C_2 + C_3}{4C^{1/4}} \ln\left(\frac{C_3^{1/4} + y}{C_3^{1/4} - y}\right) + C_1 \ln(y^4 - C_3) - \frac{4C_2 + C_3}{4C_3^{1/4}} \arctan(C_1^{-1/4} y)$$
$$+ \int \frac{y^2(y^4 - 4C_1 y + 4C_2)}{(C_3 - y^4)(y^4 - (4C_1 + 2)y + 4C_2)}\, \mathrm{d}y \tag{42}$$

where w_0 is an arbitrary constant. The integral in (42) can be evaluated. But its expression depends on the roots of $\lambda^4 - (4C_1 + 2)\lambda + 4C_2 = 0$.

6. Conclusion

In this paper we have suggested a new approach to the integration of Kramer's equations. As a result, we have reobtained known solutions as well as new ones. We briefly discuss their matching to the Levi-Civita metric. In addition we propose a solution to the longstanding problem of finding static cylindrically symmetric perfect fluid source with constant energy density. For this problem, we provide an asymptotic solution under the assumption that the pressure is weak within the source.

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